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DOUBLE-DIFFUSIVE COUNTERBUOYANT BOUNDARY LAYER IN LAMINAR NATURAL CONVECTION*

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NOMENCLATURE

c, $= [(\beta_t|T_w - T_\infty| + \beta_c|C_w - C_\infty|)g/4v^2]^{1/4}$;
C, species concentration;
f, $= FPr^{3/4}$;
F, $= \psi x^{-3/4}(4vc)^{-1}$;
Le, Lewis no., $= \alpha_t/\alpha_c$;
Pr, Prandtl no., $= v/\alpha_t$;
Sc, Schmidt no., $= v/\alpha_c$;
T, temperature;
u, v, velocity components;
x, y, Cartesian coordinates.

Greek symbols

α_t, α_c, diffusivity (thermal, species);
β_t, β_c, expansion coefficient (thermal, species);
Γ, relative buoyancy $= |\beta_t \Delta T/\beta_c \Delta C|$;
η, similarity variable $= \xi/Pr^{1/4}$;
θ, $= (T - T_\infty)/(T_{wall} - T_\infty)$;
v, viscosity;
ξ, similarity variable $= cyx^{-1/4}$;
ϕ, $= (C - C_\infty)/(C_{wall} - C_\infty)$;
ψ, stream function.

INTRODUCTION

IN DOUBLE-diffusive (or more specifically, thermosolutal) natural convection, the driving density gradients result both from temperature gradients and from concentration gradients of one or more chemical species. The associated problem of heat, mass, momentum and species transport has been studied in several classical applications [1] and is of current interest in convective stratification of magma chambers and solution mining of salt cavities for crude oil storage, Often, the Rayleigh number is sufficiently large that a double-diffusive boundary layer flow develops on internal or external surfaces, and this boundary layer will remain laminar for at least some length of the run.

The present analytical and numerical study reveals that a double-diffusive counterbuoyant boundary layer may possess the classical self-similar structure first investigated by Gebhart and Pera [2] only within two distinct and disconnected subdomains of the physical parameter space. In the outer-dominated subdomain, it is the more diffusive (and hence, outermost) of the buoyancy mechanisms which controls the primary direction of flow and the direction of boundary layer growth, and the converse is true for inner-dominated flows. Between the inner-dominated and outer-dominated domains there lies a still-unexplored domain of non-similar flows. Along the similar–non-similar borders there are marginal zones of non-uniqueness in which a multiplicity of self-similar solutions may exist. The self-similar sub-domains are investigated by several numerical methods (shooting, finite differences and Picard's method) and by the method of matched asymptotic expansions (at high Prandtl number and/or high Lewis number) in order to demonstrate the above-outlined qualitative features and to map out the borderlines of incipient counterflow. More complete discussion and results are available in [3].

ANALYSIS

The following form of the boundary layer equations expresses the conservation of mass, momentum, energy, and chemical species for a binary fluid/solute flow along an impermeable vertical wall [2]

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = v\frac{\partial^2 u}{\partial y^2} + g\beta_t(T - T_\infty) + g\beta_c(C - C_\infty) \tag{1}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha_t\frac{\partial^2 T}{\partial y^2}$$

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = \alpha_c\frac{\partial^2 C}{\partial y^2}.$$

By the introduction of dimensionless variables defined in the Nomenclature, the conservation equations are reduced to the following system of ordinary differential equations in which primes are derivatives with respect to the similarity variable ξ and $\Gamma = \beta_t|T_w - T_\infty|/\beta_c|C_w - C_\infty|$

$$F''' + 3FF'' - 2F'^2 = \mp\frac{\Gamma}{1+\Gamma}\theta \pm \frac{1}{1+\Gamma}\phi$$
$$\theta'' + 3Pr\,F\theta' = 0 \tag{2}$$
$$\phi'' + 3Sc\,F\phi' = 0$$

subject to the boundary conditions

$$\theta(0) = \phi(0) = 1; \quad F'(0) = F(0) = 0$$
$$\theta(\infty) = \phi(\infty) = 0; \quad F(\infty) = 0. \tag{3}$$

Since the roles of θ and ϕ are entirely interchangeable; we need only consider the case of $Le = Sc/Pr > 1$.

Three different transverse length-scales, or boundary layer thicknesses, are generally present: the concentration layer δ_c, the thermal layer δ_t, and the viscous layer δ_v. The ordering is $\delta_c < \delta_t$ and $\delta_t < \delta_v$ in the considered cases where $Le = \alpha_t/\alpha_c > 1$ and $Pr = v/\alpha_t > 1$.

(1) Outer-dominated flows occur when Γ is sufficiently large that the outer (here thermal with $Le > 1$) buoyancy force is dominant, and then the upper signs are appropriate in equation (2).

(2) Inner-dominated flows occur when Γ is sufficiently small that the inner (here solutal) buoyancy force is domi-

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nant, and then the lower signs are appropriate in equation (2). The signs on the buoyancy forces must change, because the primary flow direction and the direction of boundary layer growth must both be taken as positive in the direction of the dominant buoyancy force.

For a large Prandtl number, both δ_c and $\delta_t \ll \delta_v$, and in the near field buoyancy forces are in balance with shear forces. As explained by Kuiken's [4] matched asymptotic analysis, the differential equations reduce to the following near-field form in the limit as $Pr \to \infty$ (here, primes represent $d/d\eta$; see Nomenclature for f and η):

$$f''' = \mp \frac{\Gamma}{1+\Gamma}\theta \pm \frac{1}{1+\Gamma}\phi$$

$$\theta'' + 3f\theta' = 0 \tag{4}$$

$$\phi'' + 3Lef\phi' = 0$$

subject to the previously noted conditions at the wall (3a) as well as the near/far matching conditions

$$\theta \to 0, \quad \phi \to 0, \quad f'' \to 0; \quad \text{as } \eta \to \infty \tag{5}$$

which reflect the facts that the temperature (and, here also the concentration) have already fallen to the far-field value and that the f' is at a maximum (or a saddle point) at the outer edge of the near-field. Note that the near-field solution can be determined by solving (3a), (4) and (5) independently of the far-field.

The system of equations, either (2) or (4), is solved numerically using three different methods; shooting, finite differences, and Picard's method. In certain adverse circumstances, one method may perform better than the others; and the variety of approaches affords useful cross-checking to ensure that spurious numerical artifacts are not mistaken for physical phenomena. The shooting method provides the highest degree of accuracy, but it requires a very good initial guess to ensure convergence. It was implemented by library Runge–Kutta and root-finder routines, and it was often used to improve upon the solutions generated by the other methods. Picard's method of successive integrations [3], as implemented by Simpson-like quadratic integration, was found to be stable and reliable; and it was used primarily to generate the solutions to (4) for $Pr \to \infty$. The finite difference method is implemented using: the variable grid 2nd-order differencing procedure devised by Blottner [5], Newton–Raphson iteration of the non-linear product terms, and Blottner's inversion procedures for tri-diagonal and block-diagonal systems.

A key issue in the successful execution of the iterative methods (FD and Picard) is the proper selection of the parameter which is held constant during the interation cycles. Rather than fixing the relative buoyancy, Γ (for given Pr and Le), it is often advantageous to fix the shear stress $f''(0)$ or, in some high Pr cases, the velocity $f'(\infty)$ at the outer edge of the near-field. To maintain a fixed value of $f''(0)$, it is only necessary to iteratively readjust Γ in accordance with the following integral equation [e.g. for the high Pr case from (4a)]

$$\Gamma = \left(-f''(0) \mp \int_0^\infty \phi \, d\eta\right) \bigg/ \left(f''(0) \mp \int_0^\infty \theta \, d\eta\right) \tag{6}$$

based upon a prescribed value of $f''(0)$ and most recent values of θ and ϕ. An analogous integral equation is used for the finite Pr case, and likewise for maintaining a prescribed $f'(\infty)$. This utilization of an alternative parameterization of the solution family is obviously necessary when multiple solutions arise; and it may also be advantageous when other, more subtle, convergence problems are encountered.

ILLUSTRATIVE RESULTS

As an illustrative family of solutions consider the case of $Le = 4$ and $Pr \to \infty$ (hence, equations 3a, 4, 5) for all values of the relative buoyancy $\Gamma = |\beta_t \Delta T / \beta_c \Delta C|$. For $\Gamma \to \infty$, the flow is exactly the same as single-diffusive thermal convection [4]. As Γ is decreased, the counterbuoyant effect due to the solute concentration becomes progressively stronger, particularly in the wall region ($y < \delta_c$) where the counterbuoyant force is active, as apparent in the velocity profiles given in Fig. 1. The shear stress vanishes at the wall for $\Gamma = 0.63$, and as seen in Figs. 1 and 2 there is back-flow in the wall region when Γ is slightly less than this value. There is a minimum value of Γ for which outer-dominated self-similar solutions can exist, and there are multiple solutions in that neighbourhood. On the unexpected lower branch of the outer-dominated solutions which can be reached by specifying the shear stress rather than Γ, the inner counterbuoyant force exercises more influence (and hence there is more back-flow) than is observed for the upper-branch flows with identically the same counterbuoyancy ratio, Γ.

Inner-dominated flows occur for small values of Γ, since the inner solutal buoyancy is then dominant. For $\Gamma \to 0$ the flow is identically the same as single-diffusive solutal convection. As Γ is increased, the counterbuoyant effect of the thermal gradient becomes progressively stronger, particularly in the outer region ($y < \delta_c$) where only the counterbuoyant thermal force is active, as seen in the velocity profiles of Fig. 3. The velocity, f', tends toward negative values in the outer region when Γ becomes sufficiently large, and a loop-like multiplicity of solutions is found to exist in the neighbourhood of incipient back-flow (see Figs. 2 and 3) just as in the outer-dominated flows. Also, as before, there is an extremal value of Γ (here a max Γ) for which inner-dominated flow may exist.

The qualitative results are essentially the same for other Pr and Le, except for the following distinction. When Pr is finite, it is possible to have inner-dominated solutions with weak backflow in the far field. But, as $Pr \to \infty$, it is impossible, within the context of self-similar theory, for an inner-dominated back-flow to occur. The supportive argument given in [3] is based on the observations that $f'(\infty)$ cannot be negative, and that f' cannot pass through negative values and then return to zero as $\eta \to \infty$.

The limits of incipient counterflow are displayed in Fig. 4 for $Pr = 1, 10, \infty$ and for $1 < Le < 10^3$. For any Pr, Γ must either lie above the upper back-flow line or below the lower back-flow line in order for unidirectional solutions to exist. As $Le \to 1$, the problem degenerates into the mathematical

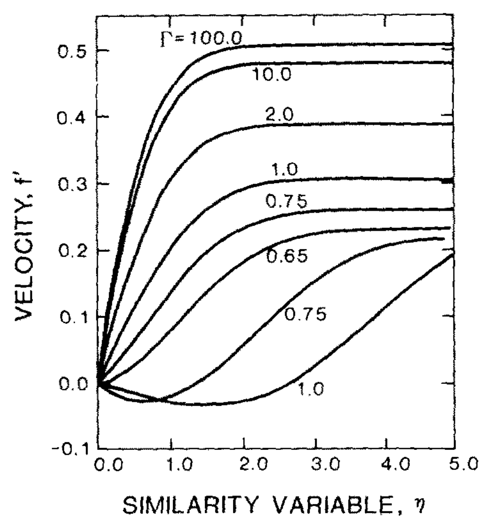

FIG. 1. Outer-dominated flows for $Pr \to \infty$ and $Le = 4$, for various values of the relative buoyancy Γ, illustrating multiplicity of solutions for $\Gamma = 0.75$ and 1.0.

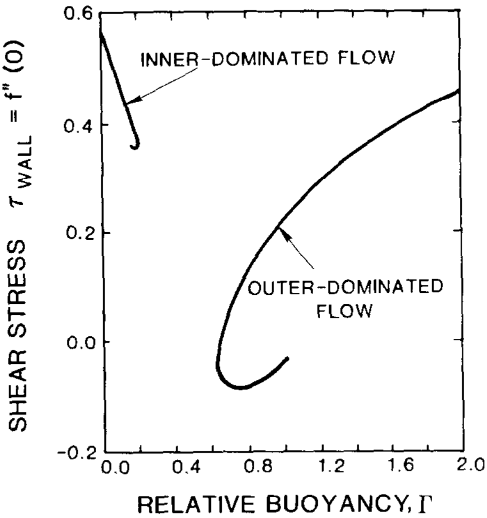

FIG. 2. Wall shear vs relative buoyancy illustrates non-uniqueness and limits of existence for $Pr \to \infty$ and $Le = 4$.

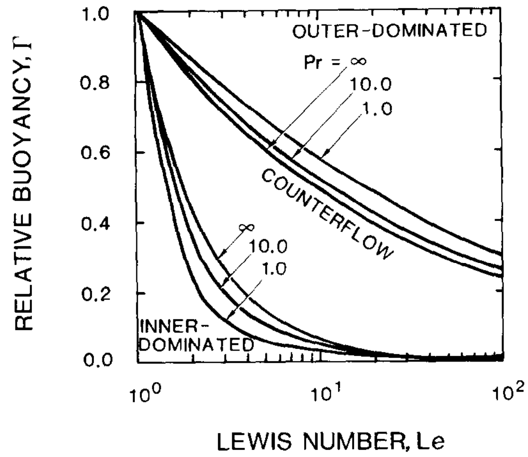

FIG. 4. Limits of incipient counterflow for various Pr.

equivalent of a single-diffusive flow; and unidirectional solutions exist for all Γ, except when $\Gamma = 1$ and there is no motion at all. The asymptotic behavior of the incipient counterflows, (in the limit as $Pr \to \infty, Le \to \infty$) is described in [3] by a three-layer model (inner/near, outer/near, and far) which divides Kuiken's near field into two regions, an inner/near zone which spans δ_c and an outer/near zone which spans δ_t, while retaining Kuiken's far-field scaling. For the inner/near zone, the scaling $f^* = f\,Le^{3/4}$ and $\eta^* = \eta/Le^{1/4}$ removes all parameters from the equations.

(1) For the outer-dominated flows there is a region of constant and non-zero shear which joins the inner/near and outer/near zones, and hence the appropriate outer/near scaling must be $\hat{f} = f Le^{1/12}$ and $\hat{\eta} = \eta/Le^{1/12}$ in order that the matching conditions can take the physically expected shear-matching form and also that the outer/near equations become dependent upon only one parameter, $\Gamma Le^{1/3}$. From the numerical computation, we find that $\Gamma Le^{1/3} = 1.08$ at incipient counterflow [i.e. when $f''(0) = 0$].

(2) For inner-dominated flows the velocity rises rapidly to a maximum within δ_c and then falls off slowly within δ_t, which suggests that the outer/near scaling must be $\hat{f} = f Le^{1/4}$ and $\hat{\eta} = \eta/Le^{1/4}$ in order that the matching conditions can take physically expected velocity-matching form and also that the outer/near equations become dependent upon only one parameter, ΓLe. From the numerical computation we find that $\Gamma Le = 0.545$ at incipient counterflow (i.e. when $f'(\infty) = 0$).

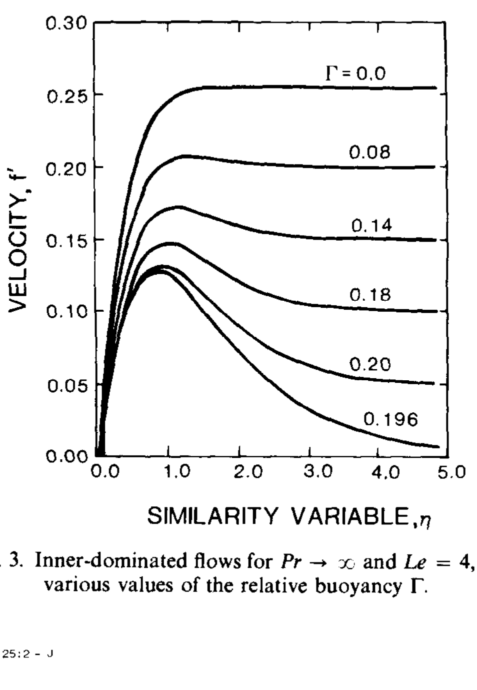

FIG. 3. Inner-dominated flows for $Pr \to \infty$ and $Le = 4$, for various values of the relative buoyancy Γ.

Incipient counterflow does not occur when the overall buoyancy forces are in balance (i.e. when $\Gamma\delta_t/\delta_c \sim \Gamma Le^{1/2} \sim 1$) as one might expect. Instead, there is a somewhat more complex interaction between buoyancy and shear forces; and this interaction is qualitatively different in the distinct cases of inner-dominated and outer-dominated flows.

SUMMARY

The flow regime map of Fig. 4 (and the asymptotic analysis) divides the parameter space into regions of: outer-dominated unidirectional flow, inner-dominated unidirectional flow, and counterflow. It has been found that the classical self-similar theory is not applicable in the central region of the parameter space where the strongly reversed counterflows are expected to occur, perhaps because the forward flow and reverse flow regions of the boundary layer may each tend to grow thicker in their respective directions of flow.

The observed multiplicity of solutions is instructive from a mathematical/numerical viewpoint, and it serves to identify the boundaries of the self-similar domain; such knowledge is useful even though stability considerations may preclude the physical occurrence of multiple solutions.

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